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A geometric approach to two-dimensional finite strain compatibility: interpretation and review

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Abstract—The purpose of this paper is to collect, clarify, augment and modify the authors' previous work on the subject of finite strain compatibility. The derivations of the fundamental equations are reviewed so that the geometric meaning of each step can be explained. Besides providing a basis for the geological interpretations of the equations, these derivations also lead to a useful new form of the strain compatibility equations.

We begin by showing that compatibility is a geometric property of continuous and smooth coordinate grids, and we derive and explain the coordinate grid compatibility equations. We then use the fact that every finite deformation may be described by two coordinate grids to derive finite strain compatibility equations in principal coordinates and Cartesian coordinates. The resulting strain compatibility equations are not easily solved for general strain fields in any coordinate system. Nonetheless, we show that many common geological strain patterns have simple geometries for which the compatibility equations can be interpreted. For example, if a deformation has constant strain in one direction, as most shear zones do, then compatibility provides an iterative method for determining the strain throughout the deformed region if the strain is initially known at any one point. Some of the other strain geometries to which we apply compatibility in this paper include simple shear, inhomogeneous pure shear, parallel and similar folding.

INTRODUCTION

ONE WAY to specify a geological deformation is by comparing two coordinate grids, one embedded in the undeformed rock and the other in its deformed counterpart. Here the word 'embedded' means that the grid lines are material lines in the rock. The derivation of the finite strain compatibility equations described in this paper is based on the assumption that these grids are continuous and smooth over some region of interest and are independent of scale. In order to interpret the strain compatibility equations, therefore, we must first understand coordinate grids, grid continuity, and how these grids relate to the material deformations which they are used to describe.

For example, the principal strain trajectories in a deformed rock form an orthogonal curvilinear grid and also form an orthogonal curvilinear grid when the deformation is removed (see Cobbold 1979 and 1980 for discussions). If these two curvilinear grids are continuous and smooth, then each must independently satisfy a set of grid compatibility equations. By relating the grid compatibility equations for the deformed state to those for the undeformed state, we can derive a single set of equations describing the continuity of the strain field. These equations are called the finite strain compatibility equations in principal coordinates. A similar approach is used to derive a set of strain compatibility equations in rectangular Cartesian coordinates.

Whether expressed in principal coordinates or rectangular Cartesian coordinates, the finite strain compatibility equations are not easily solved because they contain a large number of variables. There are several special strain geometries, however, for which one or more of these variables are zero. The following special strain geometries are considered explicitly in this paper: simple bending, uniform rotational strain, inhomogeneous pure shear, uniform area strain, uniform shape change, constant strain in one direction and simple shear. Some of the common geological structures to which these special geometries apply include parallel folds (simple bending), shear zones and similar folds (constant strain in one direction), and layer parallel shortening thrusts (inhomogeneous pure shear; terminology after Geiser & Engelder 1983).

THE COMPATIBILITY OF COORDINATE GRIDS

Orthogonal coordinate grids

By *coordinate grid* we mean a set of non-intersecting lines super-imposed on another set of non-intersecting lines such that each line in one set uniquely intersects each line in the other set. The individual lines making up a grid are defined to be infinitesimally spaced and are referred to as *grid lines*. The smallest region of space which we choose to consider and which is bounded on all

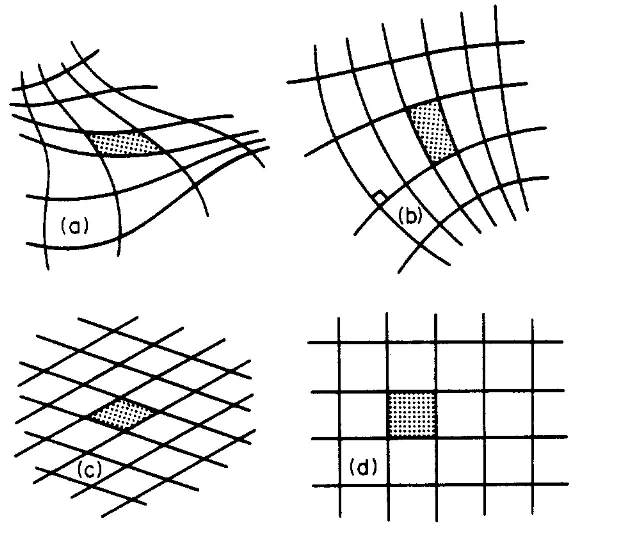

Fig. 1. Different types of coordinate grids: (a) general curvilinear (b) orthogonal curvilinear (c) oblique Cartesian (d) rectangular Cartesian. In each grid, a single grid element is shown stippled.

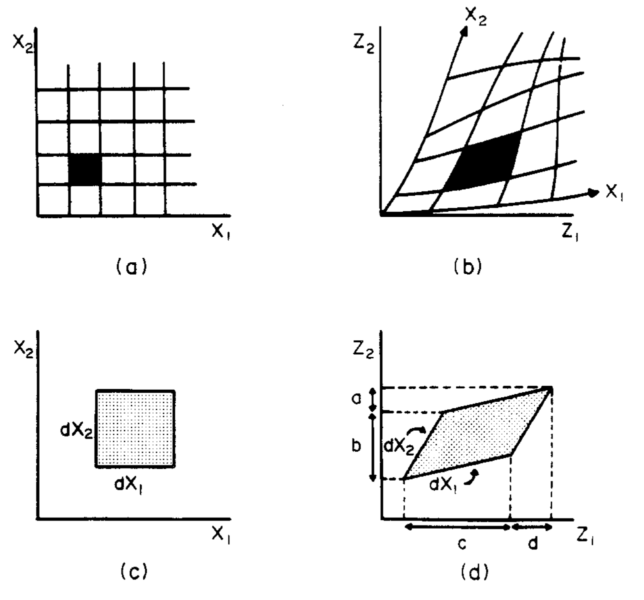

Fig. 2. The distortion of any grid (a) can be described in terms of a common Cartesian grid, as shown in (b). A single grid element is shown shaded. The infinitesimal grid element (c) distorts to a parallelogram (d). In terms of the common reference frame in (d) we have $a = (\partial Z_2/\partial X_1)\,\mathrm{d}X_1$, $b = (\partial Z_2/\partial X_2)\,\mathrm{d}X_2$, $c = (\partial Z_1/\partial X_1)\,\mathrm{d}X_1$, and $d = (\partial Z_1/\partial X_2)\,\mathrm{d}X_2$.

sides by grid lines is called a *grid element*. Grid elements are approximately parallel-sided due to their infinitesimal size.

In a general curvilinear grid, the grid lines are not straight and do not intersect at right angles (Fig. 1a). If the grid lines are not straight, but do intersect at right angles everywhere, then we call this an orthogonal curvilinear grid (Fig. 1b). If the grid lines are all straight lines, then the grid is said to be rectilinear. A Cartesian grid is a special kind of rectilinear grid which has the same spacing between grid lines, measured in some standard unit of measure such as centimeters, along each of its axes (McConnell 1957, pp. 36–40). Cartesian grids may be oblique or rectangular (Figs. 1c & d), but will be assumed to be rectangular unless otherwise stated.

The classical way to describe curvilinear grids is by means of mathematical transformations from Cartesian grids. This is normally done using the methods of tensor analysis, which require definition of covariant, contravariant and mixed tensor components (see Ericksen 1960, Truesdell & Toupin 1960, Hobbs 1971). The ensuing mathematical complexity can be avoided, at least for the purposes of this paper, by viewing all curvilinear grids as distorted versions of some common Cartesian grid. All tensor and vector components can then be viewed as Cartesian in this common reference frame, and the distinction between covariant and contravariant components disappears. It is important to note that the distortion of a coordinate grid is only related to a material deformation if we state that this is the case.

Distorted grids

To see what we mean by grid distortion, consider a curvilinear grid, X (Fig. 2b), which we imagine to be obtained by distorting an initially Cartesian grid (Fig. 2a). To describe the distorted grid, X, set up a new Cartesian grid, Z, which shares the same origin and orientation as the original grid (Fig. 2b). We call Z the common reference frame.

The distortion of grid X, if it is continuous and smooth, can be described mathematically using the transformation

$$X_1 = F_1(Z_1,Z_2) \quad X_2 = F_2(Z_1,Z_2), \qquad (1)$$

where F_1 and F_2 are independent, single-valued functions. Under very general conditions, we can invert (1) and obtain

$$Z_1 = F_1'(X_1,X_2) \quad Z_2 = F_2'(X_1,X_2), \qquad (2)$$

where F_1' and F_2' are also independent and continuous functions.

Now consider an infinitesimal element of the curvilinear grid (Fig. 2d). This grid element approximates a parallelogram and is obtained by the distortion of an initially rectangular grid element (Fig. 2c). Any element of arc can be described either in terms of the distorted grid, X, or the common Cartesian grid, Z. In terms of the distorted X grid, an element of arc will have components $\mathrm{d}X_1$ and $\mathrm{d}X_2$, while in terms of the common Cartesian frame the same element of arc will have components $\mathrm{d}Z_1$ and $\mathrm{d}Z_2$. The relationship between these two sets of components is a linear one

$$\mathrm{d}Z_i = \frac{\partial Z_i}{\partial X_j}\,\mathrm{d}X_j \qquad (3)$$

where the matrix components $\partial Z_i/\partial X_j$ are *transformation gradients* with simple geometric meanings (Fig. 2d). Viewed in terms of a Cartesian reference frame, the transformation gradients are components of a Cartesian tensor. In classical tensor analysis, however, they would be components of a tensor with mixed covariant–contravariant components.

The length, dS, of an element of arc in the Z grid is given by

$$dS^2 = dZ_1^2 + dZ_2^2 = dZ_k\, dZ_k. \tag{4}$$

To obtain the equivalent length in the X grid, simply substitute (3) into (4) to get

$$dS^2 = G_{ij}\, dX_i\, dX_j, \tag{5}$$

where the

$$G_{ij} = \frac{\partial Z_k}{\partial X_i}\frac{\partial Z_k}{\partial X_j} \tag{6}$$

are components of the metric tensor. Here the G_{ij} are considered to be Cartesian components, although in the classical theory they would be covariant components.

The polar decomposition theorem states that the matrix of transformation gradients (3) can be decomposed into a pure shape change followed by a pure rotation (Ericksen 1960, pp. 840–842, Truesdell & Toupin 1960, p. 274). An infinite number of other decompositions are also possible, but the advantage of this particular decomposition is that its components represent simple geometric operations

$$\begin{bmatrix} \frac{\partial Z_1}{\partial X_1} & \frac{\partial Z_1}{\partial X_2} \\ \frac{\partial Z_2}{\partial X_1} & \frac{\partial Z_2}{\partial X_2} \end{bmatrix} = \begin{bmatrix} H_{11}\cos A - H_{21}\sin A, & -H_{22}\sin A + H_{12}\cos A \\ H_{11}\sin A + H_{21}\cos A, & H_{22}\cos A + H_{12}\sin A \end{bmatrix}$$
$$= \begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{7}$$

where H is symmetric ($H_{12} = H_{21}$) and represents a pure shape change. The angle A is the orientation of the distorted grid element; measured as the angle between one axis of the common frame, Z, and one of the two perpendicular lines in the distorted grid element which are not rotated by the shape change component of the transformation. In order to maintain internal consistency, we always measure A as the angle (anti-clockwise positive) between the direction of maximum extension in the distorted element and the Z_1 axis of the common frame. Note the direct analogy with simple homogeneous strain, but recall that (7) represents a grid distortion and not a material deformation.

When the distorted grid is orthogonal curvilinear, then its individual grid elements are rectangular. In this case (followed in the next section), the symmetric part of the decomposition (7) simplifies to a diagonal matrix; the components of which are called *scale factors* (Malvern 1969, p. 643). In this case, the transformation (2) can be equivalently written

$$\begin{bmatrix} \frac{\partial Z_1}{\partial X_1} & \frac{\partial Z_1}{\partial X_2} \\ \frac{\partial Z_2}{\partial X_1} & \frac{\partial Z_2}{\partial X_2} \end{bmatrix} = \begin{bmatrix} H_1\cos A & -H_2\sin A \\ H_1\sin A & H_2\cos A \end{bmatrix} = \begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}. \tag{8}$$

where H_i are the scale factors. These scale factors are the reciprocals of the magnification factors described by Borg (1963, p. 69).

Matrix multiplication is not commutative, so the order of the product in (8) is significant. The rule is that the first transformation to occur is the one on the right, and later operations accumulate successively to the left (Truesdell & Toupin 1960, p. 246, Elliott 1972, pp. 2622–2623). In (8) the shape change is to the right of the rotation, which is referred to as right polar decomposition.

Grid compatibility

The preceding discussion is concerned only with the transformations for isolated grid elements. Nevertheless, all of the grid elements making up a continuous and smooth grid must fit together without gaps or overlaps. It follows that two adjacent grid elements must be compatible at their common boundary and the transformations (8) which describe the geometry of these grid elements cannot be independent. The functions which define the transformation for a grid element in terms of its position in the distorted grid are called coordinate grid compatibility equations.

The stipulation of grid continuity and smoothness requires that the differentials in (3) exist at every point in the region of interest, and this is the only pre-requisite to the following derivation of the coordinate grid compatibility equations. We begin the derivation with the rule of mixed second partial derivatives (Thomas & Finney 1979, pp. 629–632)

$$\frac{\partial^2 Z_i}{\partial X_j\,\partial X_k} = \frac{\partial^2 Z_i}{\partial X_k\,\partial X_j}. \tag{9}$$

Substituting the elements of the orthogonal transformation (8) into the continuity and smoothness requirement (9) gives

$$\begin{aligned} \frac{\partial}{\partial X_2}(H_1\cos A) &= \frac{\partial}{\partial X_1}(-H_2\sin A) \\ \frac{\partial}{\partial X_2}(H_1\sin A) &= \frac{\partial}{\partial X_1}(H_2\cos A) \end{aligned} \tag{10}$$

which after some reordering yields the compatibility equations for orthogonal curvilinear coordinate grids

$$\frac{\partial A}{\partial X_1} = \frac{-1}{H_2}\frac{\partial H_1}{\partial X_2},\quad \frac{\partial A}{\partial X_2} = \frac{1}{H_1}\frac{\partial H_2}{\partial X_1}. \tag{11}$$

Equations (11) were first introduced in the geologic literature by Cobbold (1980, equations 5). If an orthogonal curvilinear grid does not satisfy these equations over some region, then the grid lines making up that grid are not continuous and smooth curves in that region.

Finally, since it is possible to define linear transformations for non-orthogonal grid elements (equation 7), and since a coordinate grid does not need to be orthogonal in order to be continuous and smooth (Fig. 1a), it follows that coordinate grid compatibility must also apply to general curvilinear grids. Substituting the elements of the transformation (7) into the continuity requirement (9) and simplifying gives

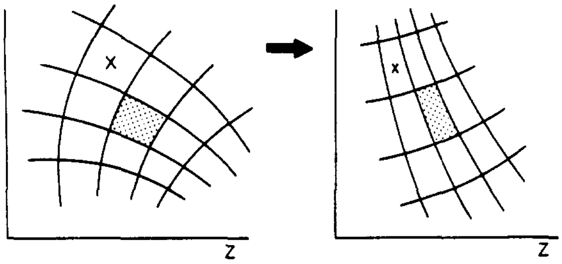

Fig. 3. Material deformations are described by the deformation of embedded coordinate grids. In order for this description to meaningfully describe the finite strain, both grids must be described relative to the common reference frame, Z.

$$\frac{\partial A}{\partial X_1} = \frac{1}{H_A}\left(-H_{11}\frac{\partial H_{11}}{\partial X_2} + H_{11}\frac{\partial H_{12}}{\partial X_1} - H_{21}\frac{\partial H_{21}}{\partial X_2} + H_{21}\frac{\partial H_{22}}{\partial X_1}\right) \quad (12)$$

$$\frac{\partial A}{\partial X_2} = \frac{1}{H_A}\left(H_{22}\frac{\partial H_{22}}{\partial X_1} - H_{22}\frac{\partial H_{21}}{\partial X_2} + H_{12}\frac{\partial H_{12}}{\partial X_1} - H_{12}\frac{\partial H_{11}}{\partial X_2}\right)$$

where $H_A = H_{11}H_{22} - H_{12}H_{21}$ is the ratio of the area of one element of the common frame, Z, to the area of the distorted element in the X grid. Note that equations (12) simplify to the compatibility equations for orthogonal grids (11) when $H_{12} = H_{21} = 0$.

THE COMPATIBILITY OF FINITE STRAIN FIELDS

The compatibility equations for coordinate grids present a relationship between the shape and the orientation of grid elements which is valid anywhere in a continuous and smooth grid. In order to make the jump from grid compatibility to strain compatibility, we have to find ways of associating coordinate grids with deformed rocks. By taking two different approaches to this grid/rock relationship, we derive two equally valid sets of strain compatibility equations, one in principal coordinates and one in rectangular Cartesian coordinates. The geometry of the strain field for a particular geologic application will determine which set of equations is more useful.

Grids and material deformations

Consider an arbitrary curvilinear coordinate grid embedded in a slab of rock. If this rock is subsequently deformed, then the coordinate grid will also change shape (Fig. 3). It follows that there are two grids associated with every deformation, the grid we define in the undeformed rock and its deformed counterpart. These grids are composed of material lines, so the geometry of each grid element in the undeformed rock is linked to the geometry of its deformed counterpart by the finite strain (Cobbold 1979, p. 68). Since infinitesimal grid elements are parallel sided, the finite strain will be homogeneous on the scale of a single grid element, although it will generally vary between grid elements. The finite strain compatibility equations describe how the strain varies from one grid element to the next in a continuously and smoothly deformed material.

Since geologists rarely observe the undeformed state, the deformed state grid provides the most convenient reference frame for applying the compatibility equations. Our first set of equations is derived by considering the compatibility of a grid composed of principal strain trajectories, since this grid is orthogonal both before and after any deformation. We will then derive a second set of equally valid equations by considering the compatibility of a rectangular Cartesian grid embedded in the deformed state, and its counterpart before the deformation. In this case, the undeformed grid is not generally orthogonal, but the result is a compatibility equation which describes the strain field using simple Cartesian coordinates.

Strain compatibility: principal coordinates

For every deformation there is one orthogonal grid which can be embedded in the undeformed rock that will remain orthogonal after the deformation. This is the unique grid that deforms to become the principal strain trajectory grid in the deformed state (Cobbold 1979, p. 68). Unless discontinuities have developed in the material during deformation, both the undeformed state and deformed state grids must simultaneously satisfy the orthogonal grid compatibility equations. We may therefore subtract the orthogonal grid compatibility equations (11) for the undeformed state from the same equations as written for the deformed state. To distinguish these sets of equations, all variables referring to the undeformed state are represented by capital letters, while those referring to the deformed state are represented by small letters. Thus

$$\begin{aligned} \frac{\partial \alpha}{\partial x_1} - \frac{\partial A}{\partial X_1} &= \frac{-1}{h_2}\frac{\partial h_1}{\partial x_2} + \frac{1}{H_2}\frac{\partial H_1}{\partial X_2} \\ \frac{\partial \alpha}{\partial x_2} - \frac{\partial A}{\partial X_2} &= \frac{1}{h_1}\frac{\partial h_2}{\partial x_1} - \frac{1}{H_1}\frac{\partial H_2}{\partial X_1}. \end{aligned} \quad (13)$$

Since the spacing between grid lines is unity in both the undeformed and deformed strain trajectory grids, and since these grids represent the same set of material lines at different points in time, X_i and x_i are completely interchangeable in equation (13). The point (3,2), for example, represents the same material point in both the x_i and X_i grids. We can thus say that $\partial\alpha/\partial x_1 = \partial\alpha/\partial X_1$, even though α represents the orientation of the deformed state grid line on both sides of the equality.

Furthermore, if the orientation of the grid element in both its deformed and undeformed states is given relative to the same common frame, then the difference between these orientations will be the rigid body rotation. In other words, $\omega = \alpha - A$ where ω is the rigid rotation and α and A are the respective orientations of the deformed and undeformed grid elements. Anticlockwise rotations are taken to be positive.

We now simplify equation (13) by taking all derivatives with respect to the deformed state grid and then substituting the rigid rotation, ω, for the difference in orientations

$$\frac{\partial \omega}{\partial x_1} = \frac{-1}{h_2}\frac{\partial h_1}{\partial x_2} + \frac{1}{H_2}\frac{\partial H_1}{\partial x_2}$$
$$\frac{\partial \omega}{\partial x_2} = \frac{1}{h_1}\frac{\partial h_2}{\partial x_1} - \frac{1}{H_1}\frac{\partial H_2}{\partial x_1}. \tag{14}$$

We have chosen to express all variables in terms of the deformed state because in geology this is what we most often observe. An equally valid form of (14) can be written in terms of the undeformed state, if necessary.

Because both grids are locally Cartesian, Cutler & Elliott (1983) wrote a simple transformation of the form (8) to unstrain an element of the deformed grid directly, yielding its corresponding element in the undeformed grid. Implicit in their analysis is the fact that the scale factors, H_i, are constants everywhere in the undeformed strain trajectory grid and can thus be chosen arbitrarily. Actually, in a curvilinear grid the scale factors in any one direction are arbitrary, but once these are chosen the scale factors in the other direction are constrained by the grid compatibility equations (11). By holding the scale factors constant in both directions in the undeformed state, Cutler & Elliott (1983) implicitly required that the undeformed grid be either oblique or rectangular Cartesian. The validity of this statement may be checked by referring to equations (11). While it is mathematically possible to define the undeformed strain trajectory grid to be Cartesian, the geologic applications of this practice may be limited.

Now, the ratios of the deformed state scale factors to the undeformed state scale factors for one grid element are the principal stretches for that infinitesimal region of the grid. We will have more use for the reciprocal principal stretches, t_1 and t_2 where

$$t_1 = \frac{H_1}{h_1} \quad t_2 = \frac{H_2}{h_2}. \tag{15}$$

Substituting (15) into (14) would give the finite strain compatibility equation as derived by Cobbold (1980, equation 12).

Here we present a new form of the equations by rewriting them in terms of true distances and the curvatures of the principal strain trajectories in the deformed state. The resulting equation will be geologically more useful because strain trajectory curvatures can often be measured in rocks and because the scale factors cancel out. The first step is to substitute (15) into (14), but also re-use (11) and let $R_s = t_2/t_1$ be the axial ratio of the strain ellipse

$$\frac{\partial \omega}{\partial x_1} = \frac{\partial \alpha}{\partial x_1}\left(1 - \frac{1}{R_s}\right) + \frac{h_1}{h_2}\frac{1}{R_s}\frac{\partial \ln t_1}{\partial x_2}$$
$$\frac{\partial \omega}{\partial x_2} = \frac{\partial \alpha}{\partial x_2}(1 - R_s) - \frac{h_2}{h_1}R_s\frac{\partial \ln t_2}{\partial x_1}. \tag{16}$$

Now, curvature is defined as the change in the orientation of a line with respect to a change in arc length along that line, so the terms $\partial\alpha/\partial x_1$ and $\partial\alpha/\partial x_2$ in equations (16) do not represent curvatures. The spacing between grid lines in the x grid will generally vary with position in the grid and cannot, therefore, be simply used as standard measures of arc length. In order to convert the independent variables, x_i, in (16) so that they correspond to standard units of measure, expand (16) using the chain rule

$$\frac{\partial \omega}{\partial s_1}\frac{\partial s_1}{\partial x_1} = \frac{\partial \alpha}{\partial s_1}\frac{\partial s_1}{\partial x_1}\left(1 - \frac{1}{R_s}\right) + \frac{h_1}{h_2}\frac{1}{R_s}\frac{\partial \ln t_1}{\partial s_2}\frac{\partial s_2}{\partial x_2}$$
$$\frac{\partial \omega}{\partial s_2}\frac{\partial s_2}{\partial x_2} = \frac{\partial \alpha}{\partial s_2}\frac{\partial s_2}{\partial x_2}(1 - R_s) - \frac{h_2}{h_1}R_s\frac{\partial \ln t_2}{\partial s_1}\frac{\partial s_1}{\partial x_1}, \tag{17}$$

where s_1 and s_2 are true arc lengths measured in the x grid. Since (17) now contains changes in orientation with respect to changes in true arc length, we can make use of the curvatures of the deformed state grid lines, $k_i = \partial\alpha/\partial s_i$. The terms $\partial s_i/\partial x_i$ are the ratios of the grid line spacing as measured in true distance to the same spacing measured in grid units. If we define our common reference frame such that its spacing is unity in whatever true distance system we are using (centimeters, miles, etc.) then these ratios become the scale factors h_i (Cobbold 1980, p. 380). Appropriately substituting k_i and h_i into (17) gives

$$\frac{\partial \omega}{\partial s_1} = k_1\left(1 - \frac{1}{R_s}\right) + \frac{1}{R_s}\frac{\partial \ln t_1}{\partial s_2}$$
$$\frac{\partial \omega}{\partial s_2} = k_2(1 - R_s) - R_s\frac{\partial \ln t_2}{\partial s_1}. \tag{18}$$

Equation (18) is perhaps the simplest form of the finite strain compatibility equations in principal coordinates. Note that all of the scale factors have dropped out and that all position changes are measured in true distance. Unless the strain field itself is truly one-dimensional, we cannot eliminate ω between the two equations in (18) using the process of one-dimensionalization (Cutler & Elliott 1983, appendix 2). Examples of how we can make use of this special case are presented in the sections on simple shear and constant strain in one direction.

Strain compatibility: Cartesian coordinates

The strain compatibility equation in principal coordinates was derived by combining the grid compatibility equations for the deformed and undeformed states. Recall that we had a choice of expressing the final equation with respect to either the deformed state grid or the undeformed state grid. It follows directly that if we want to derive a strain compatibility equation in rectangular Cartesian coordinates, then either the deformed state grid or the undeformed state grid must be rectangular Cartesian.

Since we deal with the deformed state for most geological applications, we will consider the situation of a rectangular Cartesian grid embedded in the deformed rock. In general, this Cartesian grid will undeform to become a non-orthogonal curvilinear grid. From equation (11) we see that the deformed state Cartesian grid

($\partial\alpha/\partial x_i = 0$) is continuous as long as h_1 and h_2 are constants. Setting $h_1 = c$ and $h_2 = c$ everywhere in the deformed state grid, equation (15) gives

$$ct_{ij} = H_{ij} \qquad c^2 t_a = H_A, \tag{19}$$

where t_{ij} are the components of the reciprocal stretch tensor, t_a is its second invariant and c is any constant. It is important to note that the components of the scale factor tensor are only related to the stretches by (19) because one of the grids is orthogonal. It is not in general true that $t_{ij} = H_{ij}$ for a transformation between two non-orthogonal grid elements.

Again, two sets of grid compatibility equations have to be satisfied. The undeformed state grid has to satisfy the more general equations (12), while the deformed state grid satisfies both the orthogonal grid equations (11) and the relationships (19) given above. The rest of the derivation is similar to that for the equations in principal coordinates, and involves the following steps: subtract equation (12) from equation (11) and substitute $\omega = \alpha - A$ for the rotational component of the deformation, express all of the independent variables in terms of the deformed state, substitute the relationships (19), and finally note that $\partial h_i/\partial x_j = 0$. The result of this derivation is the following set of finite strain compatibility equations referred to a rectangular Cartesian coordinate system in the deformed state

$$\begin{aligned} t_a \frac{\partial \omega}{\partial x_1} &= t_{11}\frac{\partial t_{11}}{\partial x_2} - t_{11}\frac{\partial t_{12}}{\partial x_1} + t_{21}\frac{\partial t_{21}}{\partial x_2} - t_{21}\frac{\partial t_{22}}{\partial x_1} \\ -t_a \frac{\partial \omega}{\partial x_2} &= t_{22}\frac{\partial t_{22}}{\partial x_1} - t_{22}\frac{\partial t_{21}}{\partial x_2} + t_{12}\frac{\partial t_{12}}{\partial x_1} - t_{12}\frac{\partial t_{11}}{\partial x_2}. \end{aligned} \tag{20}$$

Note that all of the constants have dropped out. Equation (20) was first derived by Cobbold (1977a, equation 7) and later rederived and discussed by Cutler & Elliott (1983, equation A11).

SPECIAL CASES OF THE FINITE STRAIN COMPATIBILITY EQUATIONS

Simple bending (parallel folding)

Parallel folds (class 1B of Ramsay 1967) occur when a layer of rock is folded such that (1) a set of initially parallel layer boundaries remain parallel after the deformation and (2) there is no shear parallel to the layering (Fig. 4a). An ideal parallel fold has a strain geometry referred to as simple bending, which occurs when one set of strain trajectories is parallel to the folded layering and the other is parallel to the dip isogons. The first application of compatibility to the simple bending geometry was by Cobbold (1980, p. 382), who used it to show that the undeformed strain trajectory grid is rectangular Cartesian. The difference between simple bending and other geometries with straight trajectories is that with simple bending, one set of strain trajectories is straight in both the deformed and undeformed states, so that the rotational gradient along all straight trajectories is zero.

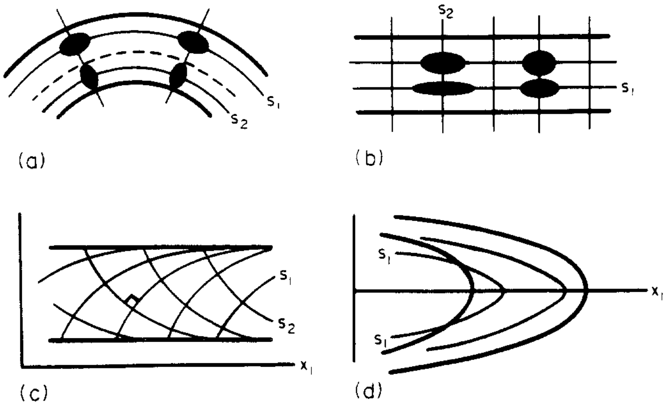

Fig. 4. The strain geometries of some geological structures for which simplified forms of the compatibility equations may be derived (a) parallel fold (b) region of inhomogeneous pure shear (c) shear zone (d) similar fold. Deformed state stretch trajectories are labelled s_1 and s_2.

In an ideal parallel fold, the geometry of the strain field changes at the neutral surface (the dashed line in Fig. 4a) as the direction of maximum extension becomes the direction of maximum compression. Here, we consider only the outer extensional region of a fold. Our results can be applied directly to the inner arc of a parallel fold by interchanging the indices and changing the signs in equations (21) and (22). In the outer arc of a parallel fold we have

$$k_2 = 0 \quad \frac{\partial \omega}{\partial s_2} = 0 \quad k_1 = \frac{\partial \omega}{\partial s_1}. \tag{21}$$

Using the fact that $0 < R_s < \infty$, the finite strain compatibility equations (18) become

$$k_1 = \frac{\partial \ln t_1}{\partial s_2} \quad 0 = \frac{\partial \ln t_2}{\partial s_1}. \tag{22}$$

In other words, the principal reciprocal stretch gradient perpendicular to the layering is equal to the curvature of that layer. Using a right-handed coordinate system and considering an antiformal parallel fold, k_1 will be negative and t_1 will increase from the outer surface toward the neutral surface. Ramsay (1967, p. 399) showed a direct relationship between the curvature of a parallel fold and the strain at some distance from the neutral surface. Note however that he assumed no area change in deriving this relationship, an assumption not contained in (21) or (22).

Uniform rotation

Uniform rotational strain, as the name implies, occurs when every grid element in some region is rotated by the same amount. In terms of the compatibility equations we have $\partial\omega/\partial s_1 = 0$ and $\partial\omega/\partial s_2 = 0$, so that the compatibility equations (18) for this case are

$$\begin{aligned} k_1(1 - R_s) &= \frac{\partial \ln t_1}{\partial s_2} \\ k_1(1 - R_s) &= R_s \frac{\partial \ln t_2}{\partial s_1}. \end{aligned} \tag{23}$$

For the most general case of deformation with uniform rotational strain, the deformed and undeformed state strain trajectory grids are both curvilinear.

Inhomogeneous pure shear

If a deformation is such that the rotation is uniform and the strain trajectories are straight in both the deformed and undeformed states (Fig. 4b), then we have $k_1 = k_2 = 0$ in addition to the uniform rotation constraint $\partial\omega/\partial s_1 = \partial\omega/\partial s_2 = 0$. In this case, the deformation is one of inhomogeneous pure shear and the finite strain compatibility equations (18) are

$$\frac{\partial t_1}{\partial s_2} = \frac{\partial t_2}{\partial s_1} = 0 \tag{24}$$

where again $0 < (R_s, t_1, t_2) < \infty$. In order to determine the continuity of an inhomogeneous pure shear deformation, we need only to measure the strain at two points on the same strain trajectory and show that (24) is satisfied. Furthermore, since only one of the principal strains is free to vary in a given principal direction, any gradient in the longitudinal strain results in an area strain gradient. Cobbold (1977b) and Cutler & Elliott (1983) reached essentially the same conclusions in their respective discussions of banded deformation structures and pure flattening, but they considered strain which varies in one direction only.

Geologically, inhomogeneous pure shear deformations are usually associated with compaction due to lithostatic loading. However, an inhomogeneous pure shear deformation may include strain gradients both parallel and perpendicular to the compaction direction. This type of inhomogeneous pure shear deformation may apply to thrust sheets which have been compacted prior to being tectonically shortened during thrusting (as described by Geiser & Engelder 1983).

Uniform area strain

In nature, area strain is difficult to measure, and the most common assumption is that there has been no area change due to deformation. The no area change assumption provides a single relationship between the two reciprocal stretches, namely $t_1 = 1/t_2$. In studies of homogeneous deformation, this relationship is often sufficient to bring about important simplifications of the governing equations. The description of general inhomogeneous deformations, however, involves too many variables for this relationship to be useful on its own.

No shape change (conformal deformation)

Now consider the case of no deviatoric strain, so that $R_s = 1$ or $t_1 = t_2 = t$. For this case it will be most beneficial to begin with the compatibility equations as expressed in (16), which now become

$$\frac{\partial\omega}{\partial x_1} = \frac{h_1}{h_2}\frac{\partial(\ln t)}{\partial x_2} \quad \frac{\partial\omega}{\partial x_2} = \frac{-h_2}{h_1}\frac{\partial(\ln t)}{\partial x_1}. \tag{25}$$

Equations (25) are the Cauchy–Riemann equations in general curvilinear coordinates. It follows directly that ω and $\ln t$ are solutions of Laplace's equation (see Appendix and Cobbold 1977a). As such, we can immediately write

$$\omega = \sum_1^\infty a_n\, e^{-nx_2} \sin(nx_1)$$
$$\ln t = \sum_1^\infty b_n\, e^{-nx_2} \sin(nx_1), \tag{26}$$

where the constants of integration are chosen to satisfy the boundary conditions in each case. Furthermore, if ω and $\ln t$ satisfy (26) over some closed region, then they must have their maximum and minimum values on the boundary of that region. The case of no shape change is subsequently one of the most potent restrictions that we can place on a strain field, although geologically relevant applications for this strain geometry may be rare.

Large strains

An interesting case of the compatibility equations occurs when the axial ratio is large. In this context, large is approximately taken to mean $20 < R_s < \infty$. If R_s is large and $\partial \ln t_1/\partial s_2$ and $\partial\omega/\partial s_2$ are not correspondingly large, then the compatibility equations (18) simplify to

$$k_1 = \frac{\partial\omega}{\partial s_1} \quad k_2 = \frac{-\partial \ln t_2}{\partial s_1}. \tag{27}$$

In other words, the extension direction strain trajectory asymptotically approaches a straight line in the undeformed state grid, which is the necessary condition for $k_1 = \partial\omega/\partial s_1$. Note the similarity between (27) and the compatibility equations for parallel folds.

Constant strain in one direction (banded deformation structures)

Much of the material in this section has been adapted or modified from Cutler & Elliott's (1983) discussion of refracted cleavage and ductile deformation zones. Constant strain in one direction means that the state of strain does not vary along a given line through a continuously deformed region, or any line parallel to that line. If the strain is constant in one direction, then all variation in the strain can be expressed in terms of one independent variable. Geological structures having this strain geometry have been referred to as banded deformation structures by Cobbold (1977b). Some geologic structures which may have this strain geometry include shear zones, refracted cleavage, some deformed stratigraphic sections, and similar folds. For our purposes, a shear zone will be defined as a zone of heterogeneous simple shear with or without a superimposed homogeneous strain.

Having established the one-dimensional nature of a deformation, the independent variables s_1 and s_2 in (18) can both be written as functions of a single position variable. The position variable we use is the angle between the direction of maximum extension and the direction of constant strain, α. This choice is valid as long as s_1 and s_2 are single valued functions of α along any monotonically increasing or decreasing segment of s_1. The uniqueness requirement makes it necessary that α always be acute and positive in this context. The reason that s_2 is a unique function of α follows from the fact that the orientations of s_1 and s_2 differ by exactly 90 degrees. In other words, all strain gradients taken with respect to α are the same, whether they are taken along s_1 or s_2.

To transform the compatibility equations (18) from two independent variables to one, begin by using the chain rule to expand the partial differentials in (18)

$$\frac{\partial \omega}{\partial \alpha}\frac{\partial \alpha}{\partial s_1} = k_1\left(1 - \frac{1}{R_s}\right) + \frac{1}{R_s}\frac{\partial \ln t_1}{\partial \alpha}\frac{\partial \alpha}{\partial s_2}$$
$$\frac{\partial \omega}{\partial \alpha}\frac{\partial \alpha}{\partial s_2} = k_2(1 - R_s) - R_s\frac{\partial \ln t_2}{\partial \alpha}\frac{\partial \alpha}{\partial s_1}. \quad (28)$$

Next, recall that the curvatures of the principal trajectories are $k_i = \partial\alpha/\partial s_i$; eliminate the rotational gradient between the two equations; and introduce the principal reciprocal quadratic stretches $\lambda'_i = t_i^2$

$$2(\lambda'_1 - \lambda'_2) = \frac{k_2}{k_1}\frac{\partial \lambda'_1}{\partial \alpha} + \frac{k_1}{k_2}\frac{\partial \lambda'_2}{\partial \alpha}. \quad (29)$$

In order to solve (29) for λ'_1 and λ'_2 we need a second relationship between these two variables. The uniform area strain assumption, that t_a^2 is a constant, provides the necessary relationship and we substitute $t_a^2 = \lambda'_2\lambda'_2$ into (29)

$$\frac{\partial \lambda'_1}{\partial \alpha} = \frac{2k(\lambda_1'^3 - \lambda'_1 t_a^2)}{(k^2\lambda_1'^2 - t_a^2)}, \quad (30)$$

where $k = k_2/k_1$. If the deformed state strain trajectory grid can be constructed, as it often can in foliated rocks, then the curvatures of these trajectories can be directly measured at any point in the deformed zone. Equation (30) can then be used iteratively to find the strain at any point in this grid, as long as the constant strain in one direction and uniform area strain criteria are met. Note that the iterative application of (30) will require initial values for λ'_1 and t_a^2, which are the boundary conditions in the problem. For most geological applications, it will be necessary to assume that $t_a^2 = 1$. Nevertheless, these boundary conditions may be evaluated at any point in the deformed region, as long as that point then becomes the starting point for the iterative process.

The compatibility equations in Cartesian coordinates can be used to gain additional insight into constant strain in one direction deformations. If x_1 is taken parallel to the direction of constant strain (Fig. 4c), then all of the strain gradients in the x_1 direction are zero and the compatibility equations (20) become

$$\frac{\partial}{\partial x_2}(t_{11}^2 + t_{21}^2) = \frac{\partial \lambda'_{11}}{\partial x_2} = 0$$
$$t_a\frac{\partial \omega}{\partial x_2} = t_{22}\frac{\partial t_{21}}{\partial x_2} + t_{12}\frac{\partial t_{11}}{\partial x_2}, \quad (31)$$

where $\boldsymbol{\lambda}' = t^2$ and $t_{12} = t_{21}$, so $\lambda'_{11} = t_{11}^2 + t_{12}^2$ (Cobbold 1977a, equation 5, Cutler & Elliott 1983, equation A7). Equation (31) says that λ'_{11} must be constant everywhere in a constant strain in one direction deformation. If the deformation is also one of uniform area strain, then the pole curve for this geometry becomes a parabola and the method of Cutler & Elliott (1983) can be used to find the Mohr circle at any point in the deformed zone. Thus, the equations in both principal and Cartesian coordinates provide ways to predict the strain as long as the area strain is uniform and boundary conditions are obtainable.

Ideal similar folds (class 2 of Ramsay 1967) have straight and parallel dip isogons (Ramsay 1967, p. 367, Elliott 1968), so that the geometry of the folded layers is constant parallel to the axial plane. Assuming that the layering was parallel prior to the deformation, the deformed layer geometry indicates that the strain must be constant parallel to the axial plane (Ramsay 1967, p. 422, Cobbold 1977b). Thus, ideal similar folds satisfy the constant strain in one direction equations and equation (30) can be used to predict the strain at any point in a similar fold with uniform area strain. Furthermore, in a similar fold oriented with x_1 parallel to the axial plane (Fig. 4d), equation (31) requires that λ'_{11} be constant in both the limbs and hinge of the fold.

A comparison of the strain geometry of a similar fold to the strain geometry of a shear zone shows that a similar fold simply represents a pair of shear zones placed back-to-back with their shear directions making up the axial plane. Since the cleavage in a shear zone can only parallel the shear direction when the axial ratio becomes infinite, it follows that the cleavage in a similar fold can only truly parallel the axial plane under equivalent circumstances. A superimposed homogeneous strain will not affect any of the above results, as the strain will still be constant parallel to the axial plane. These observations are in agreement with the geometric analysis of similar folds by Ramsay (1967, pp. 421–436).

Equivalent analyses pertain to cleavage refraction and continuously deformed stratigraphic sections. In both cases the bedding surfaces must be parallel and the deformation constant along the layering. Constant strain along the layering is generally indicated by a foliation which has a constant orientation in this direction. In the case of a deformed stratigraphic section, care must be taken that bedding plane slip is of negligible importance or the continuity assumption breaks down.

Simple shear

The geometry of two-dimensional simple shear deformations requires that (1) the boundaries of the deformed region be parallel and (2) the deformation be one of no area change. In this section we present an analytical solution to the compatibility equations (18) for the special case of simple shear. The result is a relationship between the curvatures and orientations of the principal strain trajectories. Simple-shear deformations necessarily satisfy this relationship, although every deformation which satisfies the relationship is not necessarily one of

Fig. 5. A simple shear deformation showing a single grid element in both the deformed and undeformed states. The shear direction is parallel to x_1, and A and α are the orientation of the grid element before and after the deformation respectively. The rotation is therefore $\omega = \alpha - A$.

simple shear. In fact, the relationship derived here turns out to be diagnostic of constant strain in one direction deformations, of which simple shear is a special case.

The following simple-shear equations have been adapted from Truesdell & Toupin (1960, pp. 292–295) and Treagus (1981). If deformation is known to have taken place by progressive simple shear, then the state of strain becomes a function of the orientation of the deformed state grid element, α. Consider the deformation shown in Fig. 5. If M is the material line that deforms to become the direction of maximum extension, m, then $\omega = \alpha - A$ is the rotation, $M = H_1$, and $m = h_1$. From simple trigonometry we have $M = \sec(\alpha)$, and since the simple shear model requires that $\alpha = 45 + (\omega/2) = 90 - A$, we also have $m = \operatorname{cosec}(\alpha)$. The principal reciprocal stretches become

$$t_1 = \tan\alpha \quad t_2 = \cot\alpha, \tag{32}$$

where t_2 follows from the plane strain requirement $t_1 = 1/t_2$. We can also use trigonometry and (32) to show a direct relationship between α and γ, the shear strain, as well as between α and the axial ratio R_s

$$\gamma = \cot\alpha - \tan\alpha \tag{33a}$$

$$R_s = \cot^2\alpha. \tag{33b}$$

Since simple shear is a special case of constant strain in one direction, we begin by substituting (32) and (33) into the compatibility equation (24), and then use the identities $\sec^2(\alpha) = \tan^2(\alpha) + 1$ and $k = k_2/k_1$ to simplify the result

$$k^2\tan^4\alpha - k(\tan^3\alpha - \tan\alpha) - 1 = 0. \tag{34}$$

Equation (34) must be satisfied by all inhomogeneous simple shear deformations. Now, (34) is quadratic in k and can be solved using the quadratic formula. There are two real roots to this equation, $\cot(\alpha)$ and $-\cot^3(\alpha)$, but the ratio of the curvatures of the principal trajectories can only have a single value. The meaningful root to (34) must be

$$k = \cot\alpha, \tag{35}$$

since by the definition of curvature, k, is never a negative number.

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APPENDIX

We wish to prove that ω and $\ln t$ satisfy the Laplacian in the case of a conformal deformation. The compatibility equations for conformal deformations (25) are

$$\frac{\partial\omega}{\partial x_1} = \frac{h_1}{h_2}\frac{\partial\ln t}{\partial x_2} \quad \frac{\partial\omega}{\partial x_2} = \frac{-h_2}{h_1}\frac{\partial\ln t}{\partial x_1}. \tag{A1}$$

Using the fact that second mixed partial derivatives of ω are equal, we can eliminate ω in equation (A1)

$$\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial\ln t}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial\ln t}{\partial x_2}\right) = 0. \tag{A2}$$

Exactly the same approach can be used to eliminate $\ln t$ from (25) giving the parallel result

$$\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial\omega}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial\omega}{\partial x_2}\right) = 0. \tag{A3}$$

The Laplacian equation in general curvilinear coordinates is:

$$\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial f}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial f}{\partial x_2}\right)\right] = 0, \tag{A4}$$

where f is any continuous potential function (Borg 1963 p. 80). Substituting (A1) and (A2) into (A3) gives

$$h_1h_2\nabla^2\ln t = h_1h_2\nabla^2\omega = 0. \tag{A5}$$

Finally, we note that $0 < (h_1, h_2) < \infty$ by conservation of mass, so that we can safely eliminate these terms from (A5)

$$\nabla^2\ln t = \nabla^2\omega = 0 \tag{A6}$$

which proves that both $\ln t$ and ω satisfy the Laplacian rule and are potential functions for any inhomogeneous strain field without shape change.